1. 2+4 points

Let $x_1, x_2, \ldots, x_n$ be any numbers on the interval $[0, 2]$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} |x - x_i|.$$

(In this problem, you may use fact that $|x - a|$ is continuous on $\mathbb{R}$ for any $a \in \mathbb{R}$).

(a) Evaluate $f(0) + f(2)$.

(b) Prove that there exists some number c in $[0, 2]$ such that $f(c) = 1$.

Solution.

(a) As $x_i \in [0, 2]$, we know that $|-x_i| = x_i$ and $|2 - x_i| = 2 - x_i$ for all $i = 1, 2, \ldots, n$. Hence, we find that

$$\begin{aligned} f(0) + f(2) &= \frac{1}{n}\sum_{i=1}^{n} |-x_i| + \frac{1}{n}\sum_{i=1}^{n} |2 - x_i| = \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n}(2 - x_i) \\ &= \frac{1}{n}\sum_{i=1}^{n}(x_i + (2 - x_i)) = \frac{1}{n}\sum_{i=1}^{n} 2 = \frac{1}{n} \cdot 2n = 2 \end{aligned}$$

(+2 points)

(b) Note that f is sum of continuous functions of the form $|x - x_i|/n$ for all $i = 1, 2, \ldots, n$ (the continuity from the *Hint*). Since the sum of continuous functions is also continuous, so is f. Thanks to part (a), if $f(0) = 1$ then $f(2) = 1$. This proves the statement. Otherwise, we may assume that $f(0) < 1$ since it is a symmetric argument. Again by part (a), we have $f(2) > 1$. As f is continuous, we conclude that there exists $c \in [0, 2]$ such that $f(c) = 1$ by the Intermediate Value Theorem(IVT). Therefore, the statement follows. **(+4 points)** □

Criteria

*** Part(a)**

- There is no partial point.
  - (1) If you removed absolute sign with conditions of x_i, then you get **2 pts** out of 2.
  - (2) You get **no credit** even if you wrote correct answer when you didn't remove absolute sign. In other words, used directly $|-x_i| + |2 - x_i| = 2$ without explanation.

*** Part(b)**

- You get **4 pts** out of 4 when:
  - (1) you showed that f is continuous (by using given fact) and proved three cases using IVT. Here, one is $f(0) = 1$ and the others are $f(0) > 1$ and $f(0) < 1$.
  - (2) you showed that f is continuous (by using given fact) and proved two cases using IVT. Here, one is $f(0) \leq 1$ and the other is $f(0) \geq 1$.
- You get **3 pts** out of 4 when:
  - (3) you forgot to show one of the cases in the (1) above.
- In any case, you get **(−2 points)** if you skip verifying the continuity of f when applying IVT.

2

2+7 points

(a) Assume that $\lim_{x\to 0} p(x)$ exists and nonzero. Prove that $\lim_{x\to 0} q(x)$ exists if and only if $\lim_{x\to 0} p(x)q(x)$ exists.

(b) For a positive integer n, define a function

$$f_n(x) = \begin{cases} \sin(x^n)\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find the smallest value of n in which $f_n(x)$ is differentiable at $x = 0$.
(*Hint*: You may use fact that $\lim_{x\to 0}\cos(1/x)$ does not exist).

Solution.

(a) If $\lim_{x\to 0} q(x)$ exists, then

$$\lim_{x\to 0} p(x)q(x) = \lim_{x\to 0} p(x)\lim_{x\to 0} q(x)$$

exists by Product Rule of limits. Conversely, by Quotient Rule of limits we have

$$\lim_{x\to 0} q(x) = \lim_{x\to 0}\frac{p(x)q(x)}{p(x)} = \frac{\lim_{x\to 0} p(x)q(x)}{\lim_{x\to 0} p(x)},$$

completing the proof. **(+2 points)**

(b) **Solution 1:** With the definition of differentiability, we have that f_n is differentiable at $x = 0$ if

$$\lim_{x\to 0}\frac{f_n(x) - f(0)}{x} = \lim_{x\to 0}\frac{\sin(x^n)}{x}\cos\left(\frac{1}{x}\right) = \lim_{x\to 0}\frac{\sin(x^n)}{x^n}\cdot x^{n-1}\cos\left(\frac{1}{x}\right) \tag{1}$$

exists. As $\lim_{x\to 0}\sin(x^n)/x^n = 1 \neq 0$ for all n, thanks to part (a), we know that (1) exists if and only if

$$\lim_{x\to 0} x^{n-1}\cos\left(\frac{1}{x}\right) \tag{2}$$

exists. **(+2 points)**. Thus, we have two cases:

(i) $n = 1$
Due to the *Hint*, we have $\lim_{x\to 0}\cos(1/x)$ does not exist, showing (2) does not exist. **(+2 points)**

(ii) $n > 1$
As $0 \leq |\cos(1/x)| \leq 1$, we have

$$0 \leq \left|x^{n-1}\cos\left(\frac{1}{x}\right)\right| \leq |x|^{n-1}\left|\cos\left(\frac{1}{x}\right)\right| \leq |x|^{n-1} \to 0 \text{ as } x \to 0$$

By Squeeze Theorem, we conclude that (2) is zero i.e., f is differentiable at $x = 0$. **(+2 points)**

Therefore, the smallest value of n we want is 2. **(+1 point)**

Solution 2: With the definition of differentiability, we have that f_n is differentiable at $x = 0$ if

$$\lim_{x\to 0}\frac{f_n(x) - f(0)}{x} = \lim_{x\to 0}\frac{\sin(x^n)}{x}\cos\left(\frac{1}{x}\right) \tag{3}$$

exists. Thus, we need to divide into 2 cases:

(i) $n = 1$

In this case, $\lim_{x \to 0} \dfrac{\sin x}{x} = 1 \neq 0$. Take $p(x) = \sin x / x$ and $q(x) = \cos(1/x)$. Thanks to the *Hint*, we have $\lim_{x\to 0} q(x)$ does not exist. Hence, (3) does not exist by part (a). **(+2 points)**

(ii) $n > 1$

In this case, due to Product Rule of limits

$$\lim_{x\to 0} \frac{\sin x^n}{x} = \lim_{x\to 0} \frac{\sin x^n}{x^n} x^{n-1} = \lim_{x\to 0} \frac{\sin x^n}{x^n} \lim_{x\to 0} x^{n-1} = 1 \cdot 0 = 0 \quad \textbf{(+2 points)}$$

As $0 \leq |\cos(1/x)| \leq 1$, we have

$$0 \leq \left| \frac{\sin(x^n)}{x} \cos\left(\frac{1}{x}\right) \right| \leq \left| \frac{\sin(x^n)}{x} \right| \left| \cos\left(\frac{1}{x}\right) \right| \leq \left| \frac{\sin(x^n)}{x} \right| \to 0 \text{ as } x \to 0$$

By Squeeze Theorem, we conclude that $\lim_{x\to 0} f_n(x)/x = 0$ i.e., f is differentiable at $x = 0$. **(+2 points)**

Therefore, the smallest value of n we want is 2. **(+1 point)**

□

Criteria

* Part(a)

- You need to verify which Rules are used to prove each direction. Be careful when you use limit rules. If you prove both directions at once, you get **no credit**. The reason why the equality in limit laws holds when all limits exist or diverge to infinity. But in this problem, we also consider when the limit oscillates.

  In other words, you get **no credit** when:

  (1) you used $\lim_{x\to 0} p(x)q(x) = \lim_{x\to 0} p(x) \lim_{x\to 0} q(x)$ without using existence of $\lim_{x\to 0} q(x)$. In fact, the converse of Product Rule cannot hold in general.

* Part(b)

- You get **(−2 points)** if you skip checking $\lim_{x\to 0} \sin x^n / x^n = 1 \neq 0$.
- You must use part (a) when you argued existence of $\lim_{x\to 0} x^{n-1} \cos(1/x)$. Otherwise, you get **no credit**. The common mistake includes: since $\lim_{x\to 0} \cos(1/x)$ does not exist, $\lim_{x\to 0} \sin(x^n)/x = 0$.
- Be careful when you use limit rules. As I mentioned, the converse of Product Rule cannot hold in general. So, you get **no credit** if you used Product Rule without any assumption or verification. In other words, you cannot use the following in general:

$$\lim_{x\to 0} \frac{\sin(x^n)}{x^n} \cdot x^{n-1} \cos\left(\frac{1}{x}\right) = \lim_{x\to 0} \frac{\sin(x^n)}{x^n} \lim_{x\to 0} x^{n-1} \cos\left(\frac{1}{x}\right)$$

- In any case, you need to verify non-trivial limits by using Squeeze Theorem. Otherwise, you get **no credit** in that part.

- You don't need to compute derivative of $f_n(x)$ for $x \neq 0$. Some of students use derivative of $f_n(x)$ and take limit as $x \to 0$. Note that $\lim_{x\to 0} f_n'(x) = f_n'(0)$ means f_n is continuously differentiable at $x = 0$ not only differentiable at $x = 0$. In fact, this is valid for $n > 2$.
- Also, the correct answer is an **additional point**.

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