Unit 4

Determinants and their properties

The determinant is a scalar value that can be computed from the elements of a square matrix

The determinant of a square matrix of order 2 is defined as follows.

For a 2×2matrix A defined as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant, denoted as det(A) or |A|, is calculated using the formula

$$\det(A) = ad - bc$$

Example

Find the determinant of A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution

Using the definition, det(A) = 1(4) − 2(3) = 4 − 6 = −2

Minors and cofactors of elements of matrices

the **minor** of an element refers to a specific determinant that is derived from the original matrix.

Consider the 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The minor M11 for the element a11 is the determinant of the matrix formed by removing the first row and first column:

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

Cofactor of elements of matrices

For a given square matrix A, the cofactor **Cij** of the element **aij** (the element in the i-th row and j-th column) is defined as:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Mij is the minor of the element aij which is the determinant of the submatrix obtained by deleting the i-th row and j-th column from A

Example

Consider the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}$$

To find the cofactor C11 (the cofactor of the element in the first row, first column, which is 1):

1. **Form the Minor**: Remove the first row and first column:

$$M_{11} = \begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix}$$

2. **Calculate the Determinant**:

$$M_{11} = (4)(6) - (5)(0) = 24$$

3. **Apply the Sign Factor**:

$$C_{11} = (-1)^{1+1} M_{11} = 1 \cdot 24 = 24$$

Determinants of matrices of order 3

To calculate the determinant of a 3×3matrix using the first row expansion (also known as cofactor expansion), we follow a systematic approach. Given a matrix A of the form:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \det(A) = a \cdot C_{11} + b \cdot C_{12} + c \cdot C_{13}$$

Example Calculation

Let's calculate the determinant for a specific 3×3 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

1. **Calculate the Cofactors**:
   - $C_{11} = 5 \cdot 9 - 6 \cdot 8 = 45 - 48 = -3$
   - $C_{12} = -(4 \cdot 9 - 6 \cdot 7) = -(36 - 42) = 6$
   - $C_{13} = 4 \cdot 8 - 5 \cdot 7 = 32 - 35 = -3$
2. **Calculate the Determinant**:

$$\det(A) = 1(-3) + 2(6) + 3(-3) = -3 + 12 - 9 = 0$$

Cofactor expansion of determinant along any row

The cofactor expansion of a determinant can be performed along any row (or column) of a square matrix. This technique allows us to compute the determinant by breaking it down into smaller determinants, known as minors.

Let $A = (a_{ij})_{3\times3}$ be a square matrix of order 3. Then, the determinant of A,

det(A), can be expressed as a cofactor expansion along any row of A.

That is, $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$ for each $i = 1, 2, 3$.

Example

Compute the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 12 & 0 & 0 \\ -5 & 11 & 0 \end{pmatrix}$

Solution

As the second row has two zero entries, more zero entries than the other two rows, you can use a cofactor expansion along the second row.

$$\begin{vmatrix} 1 & 2 & 1 \\ 12 & 0 & 0 \\ -5 & 11 & 0 \end{vmatrix} = -12\begin{vmatrix} 2 & 1 \\ 11 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ -5 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ -5 & 11 \end{vmatrix}$$

$$= -12\begin{vmatrix} 2 & 1 \\ 11 & 0 \end{vmatrix} = (-12)\times(0-11) = 132$$

Therefore, the determinant of A is 132.

Properties of determinants

1. If A is a diagonal matrix, then A is a triangular matrix and its determinant is the product of its diagonal entries.

   That is, if $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}$, then $|A| = a_{11}a_{22} \dots a_{nn}$

2. For a positive integer n the matrix $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$ is the identity matrix for matrix multiplication and it is a diagonal matrix.

   Thus, $|I_n| = 1 \times 1 \times \dots \times 1 = 1$.

Determinant and elementary row operations

Elementary row operations are fundamental techniques used in linear algebra for manipulating matrices.

Elementary Row Operations

There are three types of elementary row operations:

1. **Row Swap**:

Exchanging two rows of a matrix.

Effect: Swapping any two rows of a matrix changes the sign of the determinant.

If B is obtained by swapping rows i and j of A, then det(B)=−det(A).

2. **Row Scaling**

Multiplying all entries of a row by a non-zero scalar.

Effect: Multiplying all entries of a row by a non-zero scalar k multiplies the determinant by k.

Mathematical Expression: If B is obtained by multiplying row i of A by k, then det(B)=k·det(A)

3. **Row Addition:**

Adding a multiple of one row to another row.

- Effect: Adding a multiple of one row to another row does not change the determinant.

Mathematical Expression: If B is obtained by adding k·(row j) to row i of A, then det(B)=det(A).

Determinant of product of matrices and transpose

1. Determinant of the Product of Two Matrices

For any two square matrices A and B of the same size (both n×n), the determinant of their product is equal to the product of their determinants:

$$\det(AB) = \det(A) \cdot \det(B)$$

2. Determinant of the Transpose of a Matrix

The determinant of a matrix is equal to the determinant of its transpose. This property holds for any square matrix A:

$$\det(A^T) = \det(A)$$

Determinant of inverse of an invertible matrix

For any invertible square matrix A of order n:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Inverse of square matrix of order 2 and 3

The **adjoint** (or **adjugate**) of a square matrix A is defined as the transpose of the cofactor matrix of A .

The adjoint of a square matrix $A = (a_{ij})$ is defined as the transpose of the matrix $C = (c_{ij})$ where c_{ij} are the cofactors of the elements a_{ij}. Adjoint of A is denoted by adj A, i.e., adj $A = (c_{ij})^T$.

$$A^{-1} = \frac{1}{\det(A)}\left(\mathrm{Adj}(A)\right)$$

Example

Find the inverse of $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 1 \\ -4 & 5 & 2 \end{pmatrix}$

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -1; \quad c_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ -4 & 2 \end{vmatrix} = -4; \quad c_{13} = +\begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = 8$$

$$c_{21} = -\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} = 19; \quad c_{22} = +\begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix} = 14; \quad c_{23} = -\begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 3$$

$$c_{31} = +\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = -8; \quad c_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -1; \quad c_{33} = +\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2$$

Thus, adj $A = \begin{pmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{pmatrix}$

Next, find $|A|$.

$|A| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = (-1)(-1) + (-2)(-4) + (3)(8) = 31$. Since

$|A| \neq 0$,, then A is invertible and

$$A^{-1} = \frac{1}{|A|}\mathrm{adj}(A) = \frac{1}{31}\begin{pmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{31} & \frac{19}{31} & \frac{-8}{31} \\ \frac{-4}{31} & \frac{14}{31} & \frac{-1}{31} \\ \frac{8}{31} & \frac{3}{31} & \frac{2}{31} \end{pmatrix}$$

Solutions of system of linear equations using cramers rule

Theorem 4.7 (Cramer's Rule for Two Variables)

Given the system $\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$ in two variables x and y, if $ad-bc \neq 0$, then the given system has a unique solution and it is given by $x=\dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{ad-bc}$ and $y=\dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{ad-bc}$.

Example

Use Cramer's rule, if possible, and solve the following system of linear equations:

$$\begin{cases} x+y=1 \\ 2x+y=5 \end{cases}$$

Solution

The given system in matrix form is given by:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 5 \end{pmatrix} \text{ and } \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}=1-2=-1\neq 0.$$

Thus, the system has a unique solution that is given by:

$$x=\frac{\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}=\frac{1-5}{-1}=\frac{-4}{-1}=4 \text{ and } y=\frac{\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}=\frac{5-2}{-1}=\frac{3}{-1}=-3.$$

Therefore, the solution set of the given system of linear equations is $\{(4,-3)\}$.

It is worth mentioning that, in finding the solution set of a system of n linear equations in n variables using **Cramer's Rule**, we have to evaluate $n + 1$ determinants. For systems with large number of equations, Gaussian Elimination Method is more efficient.

He application is briefly written in your text book