5 اسفند

جلسه پنجم

تعریف :

$$e^{z} = e^{x+iy} = e^{x}(\cos y + i \sin y)$$

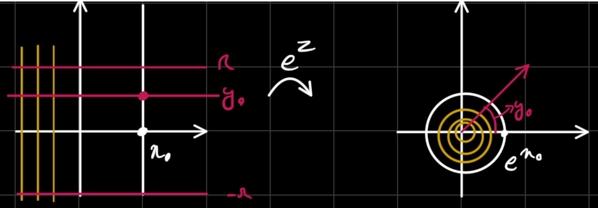
$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} \checkmark \qquad (e^{z})' = e^{z}$$

e^{z} یک تابع تام است ، یعنی روی کل $\mathbb{C}$ تعریف شده است و همه جا مشتق پذیر می باشد .

$$x=0 \Rightarrow e^{iy} = \cos y + i \sin y \quad (\text{رابطه اویلر})$$

$$|e^{z}| = e^{x} = e^{Re(z)}$$

$$\arg(e^{z}) = y = Im(z)$$



$k \in \mathbb{Z} \qquad e^{z+2k\pi i} = e^z$

$e^{\ln z} = z \qquad z \in$ دامنه لگاریتم

$$e^{\ln z} = e^{\ln(re^{i\theta})} = e^{\ln r + i\theta} = e^{\ln r}(\cos\theta + i\sin\theta)$$

$$0 < r \,,\; -\pi < \theta < \pi \qquad = re^{i\theta} = z$$

$$\ln e^z = z \Rightarrow \ln e^{x+iy} = \ln\left(e^x(\cos y + i\sin y)\right)$$

$$\ln(e^x \cdot e^{iy}) = \ln(e^x) + iy = x + iy = z \checkmark$$

$-\pi < y < \pi$

در اینجا z به پیمانه $2\pi i$ می باشد.

تعریف :

$$\cosh x = \frac{e^{x}+e^{-x}}{2} \qquad \cosh z = \frac{e^{z}+e^{-z}}{2}$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2} \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\Rightarrow \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

روابط اعداد حقیقی برای اعداد مختلط نیز صادق است

$$(\sin z)' = \cos z \qquad (\cos z)' = -\sin z$$

$\sin^2 z + \cos^2 z = 1$ $\qquad$ $\cosh z^2 - \sinh z^2 = 1$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)$$

$$= \frac{1}{2i}\left(e^{-y}.e^{ix} - e^{y}.e^{-ix}\right)$$

$$= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right)$$

$$= -\frac{1}{2}i e^{-y}.\cos x + \frac{1}{2}e^{-y}\cos x + \frac{1}{2}ie^{y}\cos x + \frac{1}{2}e^{y}\sin x$$

$$= \frac{1}{2}\sin x.\left(e^{y} + e^{-y}\right) + \frac{1}{2}i\cos x.\left(e^{y} - e^{-y}\right)$$

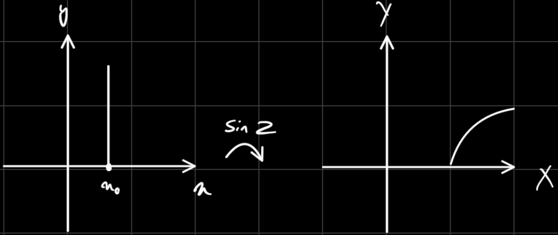
$$\Rightarrow \sin z = \overbrace{\sin x.\cosh y}^{X} + i\,\overbrace{\cos x.\sinh y}^{Y}$$

$$\Rightarrow \frac{X}{\sin x_0} = \cosh y \quad , \quad \frac{Y}{\cos x_0} = \sinh y$$

$$\Rightarrow \cosh^2 y - \sinh^2 y = 1$$

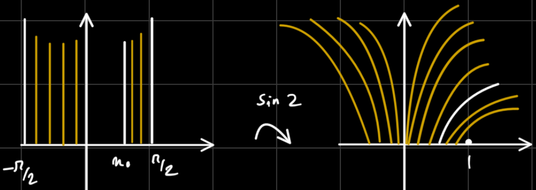
$$\Rightarrow \left(\frac{X}{\sin x_0}\right)^2 - \left(\frac{Y}{\cos x_0}\right)^2 = 1$$

هذلولی



$0 < \sin x_0 \neq 0$

$0 < \cos x_0 \neq 0$



$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

$y > 0$

$y > 0$

$$|\sin z|^2 = (\sin x . \cosh y)^2 + (\cos x . \sinh y)^2$$
$$= \sin^2 x . (1 + \sinh^2 y) + \cos^2 x . \sinh^2 y$$
$$= \sin^2 x + \sinh^2 y$$

$y = y_0 > 0$ اگر $\Rightarrow$ $\sin z = \underbrace{\cosh y . \sin x}_{X} + i \underbrace{\sinh y . \cos x}_{Y}$

$$\left(\frac{X}{\cosh y_0}\right)^2 + \left(\frac{Y}{\sinh y_0}\right)^2 = 1$$

تعریف :

$$z^c = e^{c \ln z}$$

$$c^z = e^{z \ln c}$$

مثال :

$$i^i = e^{i \ln i} = e^{i(\ln|i| + i \arg(i))}$$
$$= e^{i(0 + i(\frac{\pi}{2} + 2k\pi))}$$
$$= e^{-(\frac{\pi}{2} + 2k\pi)} \quad k \in \mathbb{Z}$$