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International Journal of Mathematical Education in Science and Technology

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/tmes20

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To cite this article: Sevda Dolapcioglu & Ahmet Doğanay (2022) Development of critical thinking in mathematics classes via authentic learning: an action research, International Journal of Mathematical Education in Science and Technology, 53:6, 1363-1386, DOI: 10.1080/0020739X.2020.1819573

To link to this article: https://doi.org/10.1080/0020739X.2020.1819573

Published online: 17 Sep 2020.

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY
2022, VOL. 53, NO. 6, 1363–1386
https://doi.org/10.1080/0020739X.2020.1819573

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Development of critical thinking in mathematics classes via authentic learning: an action research

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ABSTRACT

The aim of this study was to examine how fifth grade students' critical thinking and problem-solving skills can be developed using practices based on Newman and Weglage's (1993. Five standarts for authentic instruction. *Educational Leadership*, 50(7), 8–12) authentic learning standards (*high-order thinking*, *depth of knowledge*, *relation with the world beyond the classroom*, *constant conversion* and *social support for student achievement)*. The main aim of the study was to offer different perspectives on mathematics education by focusing on development of critical thinking skills needed to understand mathematics. The study was carried out in six cycles via an action research model. The data consisted of unconstructed observations (camera recordings), students' scores on CTR (Critical Thinking Rubric), and their journals and written documents. The research results showed that practices based on authentic learning standards improved such critical thinking skills as comprehending, comparing, proving, suggesting new solutions, and reflecting on problem solving processes of the students who were included in the target study group.

ARTICLE HISTORY
Received 25 July 2019

KEYWORDS
Mathematics; critical thinking skills; authentic learning

1. Introduction

Concerns regarding mathematics instruction are generally focused on students' knowledge of mathematics topics (knowing) and ability to find correct solutions to problems with the help of prior knowledge (doing). However, there is another component between these two objectives,to which problem-solving in mathematical learning is most related. This component is deep understanding of mathematics, which teachers neglect if they do not probe students' thinking. As Erickson (2007) suggested, traditional instructional designs that focus on students' 'knowing' and 'doing' processes are mostly two-dimensional. However, teachers of mathematics should apply techniques to develop students' critical thinking, that enable deep understanding, rather than memorization-based techniques (Peter, 2012). The term 'deep understanding' in this study implies a profound interpretation of mathematical knowledge and involves comparison, evaluation and proving of solutions, suggesting new ways of reaching solutions, and reflecting on one's learning. These skills are sub-components of critical thinking skill (Ennis, 1991; Paul, 1995).

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Critical thinking is an intellectually disciplined process that includes active employment of skills for comprehension, application, analysis, synthesis, observation, interpretation of experience, reflection, reasoning and communication (Paul, 1995), all of which are processes emphasized in mathematics learning. Mathematics learning can be described as engagement with a set of instructional topics and processes intended to help learners develop problem-solving skills, thinking skills, reasoning skills, and numeracy skills along with economy of thought (Animasaun & Abegunrin, 2017). Thus, supporting improvement of critical thinking skills through mathematics can advance students beyond merely answering questions asked and solving problems posed. Several studies have addressed this connection between understanding mathematics and developing critical thinking (Aizikovitsh & Amit, 2010; Alkhateeb, 2019; Boucher, 1998; Butera et al., 2014; Enright & Beattie, 1992; Gainsburg, 2008; Kitchener, 2008; Peter, 2012; Saleh, 2009; Schwartz, 2006; Suh, 2010). Some have also argued that there is a lack of deep mathematical understanding in mathematics teaching (Alkhateeb, 2019; Butler, 2019). On the other hand, few studies have been conducted to develop a learning model that supports actionable solutions to problems related to the development of sub-components of critical thinking in order to promote deep mathematical understanding. Such a model needs to present ways to enable important instructional strategies in detail, because the development of both critical thinking and mathematical understanding are process-based and need time. For this reason, in this period of educational research, in which positivistic research is replaced by critical and interpretive action research, an action research model can be regarded as an important contribution of the current study to the field of mathematics teaching. Another contribution of this study is that it addresses problems of learning mathematics which are common in both local and global environments, in which similar problems of mathematics are encountered, by proposing a model with a powerful theoretical infrastructure. The critical action research approach focuses on broad, national and perhaps international problems, even when presenting local problems (Willis & Edwards, 2014).

Improving critical thinking in mathematics education is a general objective of all countries (Aizikovitsh & Amit, 2010; NCTM, 2000). Moreover, several researchers contend that improvement of critical thinking skills directly impacts mathematics achievement (Aizikovitsh & Amit, 2010; Firdaus et al. 2015; Peter, 2012). Critical thinking is a complex way of thinking that includes understanding, reasoning, analyzing, problem solving, reaching the right solution, proving, and reflecting on others' solutions (Beyer, 1991; Marcut, 2005; Paul, 1995). The focus of this study was on the skills of understanding, comparing solutions, reaching the right solution, proving, suggesting new solutions, and reflecting.

Briefly, critical thinking in mathematics entails not only knowing and using knowledge for reaching correct solutions, but also understanding, interpreting, investigating different ways of finding solutions, and reflecting on the benefits of mathematics in everyday life. Students cannot find meaning and cannot develop higher-order thinking skills in learning environments in which they do activities based merely on knowledge repetition, which does not require them to use their minds, reflect on their thinking, or discuss ways of finding solutions (Newmann, 1990; Ritchhart, 2002). Activities and learning setting suitable to these activities are included in the principles of the authentic learning model. Therefore, authentic learning is a significant way for the development of critical thinking, a higher-order thinking skill (Dennis & O'Hair, 2010; Gainsburg, 2008; Preus, 2012) by providing a transition for students to progress beyond knowledge repetition. Authentic learning refers to a model for providing and applying skills for such processes as critical thinking, creative thinking, research, communication, problem-solving, entrepreneurship, decision-making, using sources effectively and self-management (Lombardi, 2007; Newmann & Weglage, 1993). In this approach, it is obligatory to plan activities that provide time for interpretation and reflection on what has been learned. The authentic learning rules in Newmann and Weglage's (1993) learning setting, highlighted in this study, promoted higher-order thinking, depth of knowledge, connection with the out-of-class world with real life problems, meaningful dialogues, and social support for student success (Boucher, 1998; Gatlin & Edwards, 2007; Makina, 2010; Preus, 2012). To address these teaching and learning goals, this study was guided by this over-arching question: How would the 'authentic learning model' affect students' development in the dimensions of

- 'understanding',
- 'comparison and evaluation of solutions',
- 'reaching the right solution and proving it',
- 'suggesting new solutions', and
- engaging in 'reflection' as critical thinking skills over time?

∗This study was conducted from 2013 to 2014, during the author's PhD studies at the Çukurova University of . . .

2. Theoretical background

Marcut (2005) explained that thinking about mathematics critically was first discussed by Harold in 1938. Toward this end, students' improvements in their critical thinking skills need to be followed through indicators while they are participating in classes. These indicators include discussing, hazarding opinions, making decisions, reflecting, clearly conveying their own mathematical thinking to the others, analyzing others' mathematical thinking, and carrying out evaluation. Enright and Beattie (1992), who also provided an important contribution to the development of critical thinking skills in mathematics, presented the SOLVE (Study the Problem, Organize the Data, Line up a Plan, Verify the Plan, and Evaluate the Match) project. In this project, the aim was to determine the critical thinking skills that need to be developed through exposure in multiple disciplines and by associating problem-solving skills used in mathematics with skills that are needed in real life. However, many researchers who emphasized the necessity of developing critical thinking skills in mathematics presented only the need for the connections without offering a concrete model. For example, Glazer (2001) stated that a web-based instructional resource was needed for the development of critical thinking skills in high school mathematics classes. Baugh (2002) expanded on the compatibility of critical thinking with the high school mathematics curriculum, and Kitchener (2008) investigated the development of critical thinking skills of ninth and sixth graders, finding that the students' abilities in teamwork and working with other people, time-management, reflective thinking, and reading comprehension improved with the help of creative thinking. Saleh (2009) suggested that a mathematics program was necessary for the development of children's thinking skills, which were of great importance in terms of adapting to society and continuing to improve.

Similarly, Suh (2010) investigated whether use of technology in elementary school mathematics would create a favourable environment for developing students' critical thinking skills. In this study, projects were designed to lead students to use technology to design and implement mathematical tasks. Additionally, Makina (2010) discussed the importance of visualization skills to the development of critical thinking skills in mathematics classes for the ninth graders. He stated that mentally visualizing objects and their original lengths was a crucial part of critical thinking and can promote its development. In their study of the integration of mathematics problem solving and critical thinking into the curriculum, Butera et al. (2014) focused on improving students' critical thinking and problem-solving skills from early ages. Peter (2012) emphasized that instructional techniques that foster active student participation in the mathematics learning process rather than memorization techniques can improve students' critical thinking skills. Aizikovitsh and Amit (2010) found in their study of the sub-dimensions of critical thinking (prediction, inference, understanding, decision-making and observation) in mathematics that exercises based on collaboration and communication developed critical thinking further than traditional methods. Butler (2019) highlighted that encouraging learners to develop their critical thinking and reflection can facilitate their comprehension and understanding in mathematics. Alkhateeb (2019) found that if an instructor intends to improve students' thinking processes, uses effective interrogation techniques, and guides students through the critical thinking process, the students can become critical thinkers.

All of the skills discussed by the researchers can develop critical thinking in mathematics, including conceptual understanding, interpretation, proving, evaluation, and reflection on mathematical knowledge. Long-term studies involving each of these dimensions need to be performed because critical thinking is a process that is intellectually disciplined, and the ability to actively and skillfully evaluate the data collected from experience with communications, observations, perceptions, practices, analysis, synthesis, reflection, logic, and reasoning (Paul, 1995) takes time. Similarly, determining goals for content, creating key questions, questioning knowledge, recognizing concepts, making inferences, admitting assumptions, taking outcomes into account and constructing a perspective (Beyer, 1991; Paul & Elder, 2001) are sub-skills of critical thinking. Consequently, there is a need for learning models that give us the opportunity to follow the development of each skill by studying it individually. The current research, therefore, was unique in that the development of each skill was monitored separately and information regarding the process was presented.

A number of studies have demonstrated that mathematics provides an opportunity for the development of critical thinking skills, and that approaches for developing critical thinking skills are effective for deeper understanding of mathematical knowledge. Thus, aim of the current study was to analyze how the authentic learning principles presented by Newmann and Weglage (1993) support the development of sub-components of critical thinking skills (Dennis & O'Hair, 2010; Gainsburg, 2008; Preus, 2012) and contribute to deep understanding of mathematics as well as how to use them to provide a model for fostering thinking skills, acknowledging that certain features of the learning setting are significant for the development of critical thinking skills. Authentic Learning is a strategy that includes higher-order thinking and real-life problems (Gatlin & Edwards, 2007; Preus, 2012). Real life problems in the authentic learning environment include working both outside of school and within the school setting for students to develop necessary knowledge, skills and attitudes (Bektaş & Horzum, 2014; Callison & Lamb, 2004; Rule, 2006; Nicaise et al., 2000). According to Cholewinski (2009), the concept of authenticity is used synonymously with 'real objects related to life used to teach real life in the classroom'. Authentic learning is an opportunity to improve educational outcomes and to promote student participation (Herrington, 2006). Additionally, one of the components of teaching for authentic learning is planning learning resources and the assigned tasks around learning goals (Herrington & Oliver, 2000; Lombardi, 2007; Parker et al., 2013). Another contribution of this study is that it was planned as action research, which focuses more on qualitative data and actual mathematical problem-solving to develop critical thinking skills. Such research involving authentic learning, which focuses on actual classroom dynamics and learning processes, can shed light on the development of critical thinking skills and help fill in the gaps in the literature. It may also provide alternatives approaches to the development of critical thinking skills.

3. Method

To investigate the effects of authentic learning practices on students' development of critical thinking skills, participatory action research, a qualitative research pattern which includes systematic data collection for analysis of an existing problem, was employed (Johnson, 2005; Kemmis & McTaggart, 2000; Mills, 2003; O'Brien, 2001). Action research is an ongoing cyclic process that includes planning, observing, acting and reflecting, intended to reshape prior ideas about a situation. The model is based on the participants' understanding of the situation according to their own experiences and implementations (Kemmis & McTaggart, 2000, p. 563; Willis & Edwards, 2014). Similarly, the current study was based upon the principles of the authentic learning model, focusing on the participant instructor's experiential learning by experience and on the student participants' improvement of their critical thinking skills. Authentic learning standards were based on scientific knowledge concerning the development of critical thinking skills in mathematics teaching (Figure 1).

The process above informed the establishment of a learning environment compatible with the authentic learning materials through six cycles in total, and the students' development of critical thinking was observed.

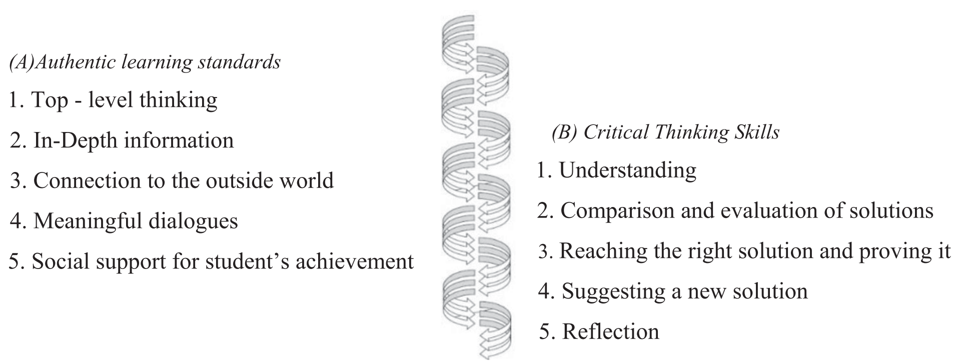

Figure 1. Action cycle process.

(A) ***Authentic learning standards***

Authentic learning is a strategy involving higher-order thinking, depth of knowledge, connection with the out-of-class world, meaningful dialogues, and social support for student success (Newmann & Weglage, 1993) as well as real-life problems (Boucher, 1998; Gatlin & Edwards, 2007; Makina, 2010; Preus, 2012). These are explicated further below.

(1) The scenarios related to the principle of higher-order thinking that were used in the study included asking students to determine the right solution between two solutions offered by proving, instead of directly figuring a solution out. The students were also asked to suggest different solutions and to reflect on their learning (see Appendix 1). They compared, analyzed and reflected on their solutions with their peers in groups of four. In this way, a thinking classroom environment was created.
(2) In relation to the principle of depth of knowledge, repetition of mathematics topics allowed students to first learn individually and then to expand their understanding in group discussion. A total of 24 reading scenarios were provided, 18 of which were for practice and 6 for evaluation. Each reading scenario was allocated one week, four lesson hours per week. The mathematical content of each scenario involved real-life examples of uses of natural numbers, numbers, fractions, percentages, and length and time measurement, based on the requirements of the standard curriculum. For each scenario text, the students first read individually during the time allocated and then shared their understandings with group members. Thus they had the opportunity to repeat the knowledge and to relisten to it from their peers.
(3) The connection with the out-of-the-class world was enhanced through the reading scenarios, which presented real-world problems that called on mathematical knowledge, and the students were given the opportunity to use their own experiences as content to apply the knowledge. Each scenario consisted of five components: understanding, comparison and evaluation of solutions, reaching the right solution and proving it, suggesting new solutions, and reflection (Appendix 1).
(4) To engage in meaningful dialogue, the students discussed the importance of speech to learning and understanding the basis of a subject. In a classroom in which there is little or no meaningful dialogue, interaction is restricted to the teacher's plan and questioning and students' short answers. In order to increase the amount of meaningful dialogue and sharing of ideas, group activities were carried out for each scenario. The participants thus had opportunities to explain themselves, to compare their solutions, and to discuss each speaker's comments. To foster co-operative understanding of a theme or topic. The dialogues were structured to promote harmony rather than contention.
(5) Social support for student success may be achieved when a respectful classroom setting is established, in which hard work is applied to academic study, risks may be taken, all classroom members gain knowledge, and all are successful. An emphasis on mutual respect supports the efforts of less-talented students, and their contributions are valued. The classroom setting in this study was structured in accordance with this principle.

(B) ***Critical Thinking Skills:***

There are several sub-skills of critical thinking skills. Our study focused on the development of the following five main sub-skills under mathematical thinking skills and their criteria:

(1) Understanding: Expressing the main idea and problem logically and consistently in one's own words and statements.
(2) Comparison and evaluation of solutions: Comparing the solutions given and using induction and deduction to accurately determine the relevant knowledge to evaluate them.
(3) Reaching the right solution and proving it: Solving the problem correctly and with the correct process steps and then accurately and clearly explaining the solution.
(4) Suggesting a new solution: Associating different information and ideas in a correct way and finding a different way to reach the right solution.
(5) Reflection: Being aware of the information learned individually and presenting it to a group with meaningful interpretations.

3.1. Participants

The study group consisted of 34 fifth graders, 18 girls and 16 boys. The students were studying at a mid-level socioeconomic status school in Hatay, Turkey. The students' critical thinking skills were evaluated using the CTR. To determine their levels, the students' scores were ranked from the lowest to the highest. Two were at the third level, five were at the second level, and two were at the first level, while the majority (23) were between the first and second levels. These findings showed that the participants experienced problems related to the relevant skills. Additionally, 16 of these students were from low socio-economic families, 14 from middle socio-economic families, and four from upper socio-economic families. The teacher in the study was male, had the bachelor's degree in Mathematics Education, and had eight years of teaching experience. He had not received any training in teaching critical thinking skills after his university education. The teacher collaborated with the researcher during the study, watched video recordings of his teaching with the researcher to critique his implementation of authentic learning standards and identified shortcomings in his teaching. He also made decisions with the researcher on ways to correct them and contributed to the development of the next stage of the research process.

3.2. Data collection tools

The data to address the research question of this study, which called for monitoring the development of students' critical thinking skills with teaching based on authentic learning standards, were collected over a period of two semesters in three phases, before, after and during the research process. The data consisted of video-recordings of students' group activities, rubrics of critical thinking skills measurement (CTR), and interviews. CTR (See Appendix 2) was administered to obtain indicators of development of critical thinking skills (Boucher, 1998; Butera et al., 2014; Ennis, 1991; Enright & Beattie, 1992; Kitchener, 2008; Konobloch, 2003; Makina, 2010; Marcut, 2005; Saleh, 2009; Schwartz, 2006; Suh, 2010) and higher-order thinking skills (Brookhart, 2010; Haladyna, 1997). The researcher watched the video-recordings of students' critical thinking-based behaviours following each implementation and wrote an account of each student's behaviours. The researcher then analyzed these written documents and determined the frequencies of the students' their critical thinking behaviours skills during group study. Means of frequencies as identified by two separate experts were calculated using the Excel program. Finally, the 'prevalence of each skill in the students' was found by summing their separate critical thinking behaviours. A rubric was used to evaluate each student's written responses to the open-ended questions. Additionally, interviews were conducted with the students and analyzed to determine their perceptions of their own development.

3.3. *Validity and reliability study in the research*

Reliability in action research depends on careful collection and unbiased analysis of data, and validity depends on creating a complete and detailed picture based on sufficient observations (Johnson, 2005). During the implementation phase of the research, a committee of three experts, one from the university's department of primary school education and two from the department of educational programming and teaching, was created to monitor the data collection and analysis processes, to give feedback to the researcher on effective and incomplete aspects of the research process, to suggest new perspectives, to discuss the problems that arose and to offer suggestions. In this context, the following procedures were carried out in order to ensure validity and reliability of the research:

- Adequate time was spent in the setting during the case observation period in order to prevent data loss during the research. Classroom observations were conducted for a period of four weeks. Data also comprised 24 weekly video recordings.
- The data were collected from four sources including video-recordings, rubrics, written journals about critical thinking skills, and recorded interviews. Thus, the data sources achieved triangulation and could be cross-checked for consistency.
- A certain number of video recordings were analyzed by another expert for comparison with the themes determined in the data analysis. The data obtained through rubric were also scored by two separate encoders, and the correlation between the scores was 81.

4. Findings

The results of the data analysis indicated that the students showed improvement in the five sub-skills of critical thinking understanding, comparing and evaluating solutions, reaching the right solution and proving it, suggesting a new solution, and reflecting on learning. These findings are discussed below in relation to two primary data sources, observations and CTR.

4.1. *Findings regarding observations of the development of critical thinking skills*

As noted, video-recordings were analyzed after each application to determine the contribution of authentic learning standards to the development of students' critical thinking skills and identify problems that might arise in practice, The development of the sub-skill of 'understanding' in its cyclical period is illustrated in Figure 2.

Understanding was determined by the extent to which the student could express and interpret main ideas of the mathematical problems in the reading scenarios in his/her

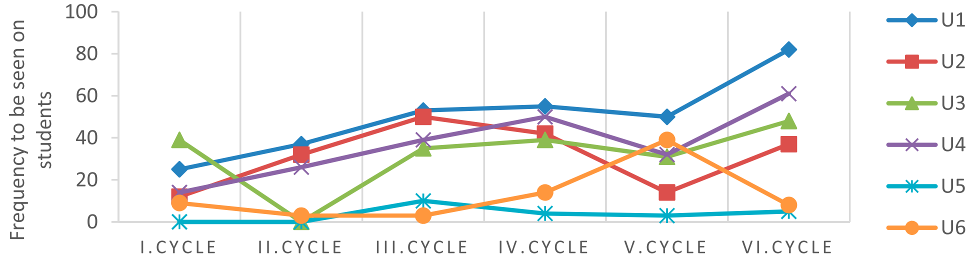

Figure 2. Findings regarding observations of the development of the sub-skill of understanding. * U Understanding.

own words. The students' in-class behaviours that might be indicators of this development were monitored during the implementation period, and their frequencies were determined. While the students read the scenarios verbatim at the beginning of the process, they started to express the problems using proper and meaningful sentences in the second cycle (U1). It was observed that as the frequencies of the behaviours increased, the students' levels of speaking (U2), reading for understanding (U3), listening to the scenario from peers (U4), learning the meanings of new words (U5) and expressing the problems with proper and meaningful sentences (U1) improved. It was also observed that the frequency of asking questions (U6) went up in the fifth cycle, in which the topics of incomplete prior knowledge were learned.

The frequency of students' U1-related behaviours was lowest in the first cycle. Analysis of the video-recordings indicated that **the students regularly sought the right solution to the problem with haste, and the teacher tended to give the content directly**, just as it happened in all other dimensions:

- *Student 12: I was reading the question in the exam and I was selecting the option that first came to my mind. Now I read the options one-by-one, try to understand, discuss with my friends if I do not know any words, and I think it. I improved my grades thanks to this . . .*
- *Student 18: I comprehended the topic very well today because I discussed the topic with my group mates, and we spent a lot of time on it.*
- *Student 14: I used to pay no attention to things around me . . . I was solving the problems by ignoring to read them but now I consider about the solution of the problems carefully, and I explain them in figures.*
- *Student 16: Teacher! I had not been able to understand length, and I had had difficulty since I had been the first grader, I had not understood it at all, but I did this year. For example, we took a walk the previous day, and there was a long way in front of us. I considered about its length in kilometers, and about how much longer it was than my height.*

The students' skills of interpreting, of explaining ways of finding solutions, and of explaining developed:

Teacher: . . . most of the students were barely uttering a few sentences at the beginning of the process but now they interpret what they have learned, they easily respond to the questions and explain the solution they believe and find right . . . just then this skill developed during the process so that they started to find words to comfortably express what they want, and they can compose and produce these words.

The development of students stood out more explicitly in time:

Teacher: It has been a year since the implementation started. Now, the students have become the ones to whom I can teach all of the topics earlier than the other 6th graders and they are the best in terms of understanding what I teach. I think this stems from our implementation. It seems like the effort we have put in can be seen more clearly as the time goes on. They did not used to understand so fast in previous times.

For the fifth cycle, the decrease in the frequency of U2 (speaking about the scenario) behaviour **might be attributed to starting a new subject and to the students' inadequate prior knowledge** because it was seen that the development of all other critical thinking skills (see Figure 7) still continued in this cycle. For example, evidence of development appeared in the behaviours of the skill 'Comparison' in the second dimension (see Figure 3). This resulted from the students' difficulties in learning the new topic, so they suggested different solutions, which they compared with those of their friends. Working patiently this way affected development in the sixth[h] cycle. On this basis, it is possible to suggest that for new topics or for topics which students may have difficulty understanding or lack prior knowledge, comparing solutions with other students in group work and explaining the reasons for their erroneous solutions can support students' development of deep understanding of mathematics. The teacher's reflection on students' development in the sixth cycle supports this assertion:

Teacher: The students' skill of understanding has improved because of long scenarios and of the fact that the students were asked to understand the problem within the scenario and to discuss it. When the students had disagreement on a matter in their group, they were able to listen to each other respectfully. The most important thing was that they explained the reasons for a wrong solution to each other, they convinced others, and they agreed on the most reasonable way of solving the problem.

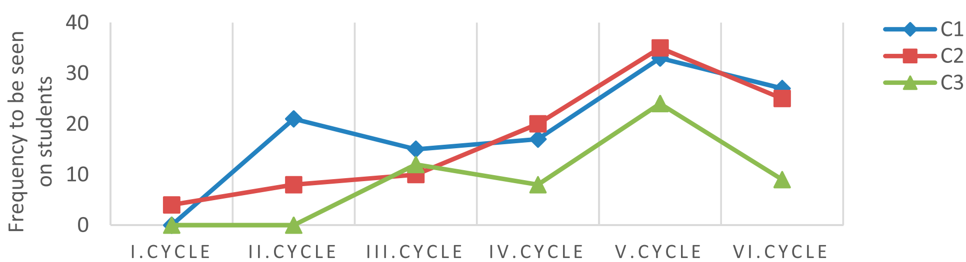

Figure 3. Findings regarding observations of the development of the sub-skill of comparison. * C: Comparison.

The principle of 'depth of knowledge' in authentic learning enabled frequent repetitions, so the teacher could discover what was not understood by the students and foster the skill of understanding:

Teacher: As I benefitted from many repetitions as required by authentic learning practices, and the students were involved in the process, I realized that there was a great lack of information even in the topics that I was sure that they had learned very well. I took this inadequateness into consideration in the action plan and worked on it.

Based on the findings obtained for this sub-skill, it can be stated that applying such principles of authentic learning as giving students time for individual thinking and engaging them in peer discussions in which they compare solutions and provide evidence for their thinking can support their development of deep understanding in mathematics. The point to be emphasized here is that it is important to create an environment in which critical thinking is valued over quickly reaching a right answer.

These findings revealed low frequencies of such behaviours as comparing solutions among themselves (C1) and explaining reasons for erroneous solutions (C3). In particular, their frequencies of analyzing each other's solutions (C2) were low as they tended to share their solutions only with their teacher. Also, the students who could not find solutions waited for the teacher's explanation or copied the right solution from the board. A reason for these low frequencies could be the teacher's difficulty in changing his habitual instructional methods and traditional classroom setting:

Teacher: As teachers, we always try to teach our lessons in front of the board just by ourselves, but the students do not learn and understand since they are not active. I noticed this problem with the help of this study. From then on, I have been paying attention to involve more students in the process during my lessons in other classes.

This finding underscores the need for further research on ways to convince teachers of the ineffectiveness of traditional teaching habits and help them develop approaches that give students the opportunity and enough time to evaluate and explain their solutions. Students showed some improvement was observed from the second cycle in examining each other's solutions, comparing their different solutions and explaining their own solutions with reasons, as they were provided with social support for solving real-life problems based on higher-order thinking. In learning a new topic in the fifth cycle, they more often compared and tried to prove their solutions:

- *Student 10: I had a great difficulty in comparing and admitting the correctness of the solutions given in the second activity.*
- *Student 3: I used to think that only my opinions were right; now I listen to my peers' opinions as well.*
- *Student 9: I didn't have much self-confidence. I always thought my answers were wrong and those of my peers were correct. However, I sometimes explain why I choose that way of solution when we compare our solutions.*

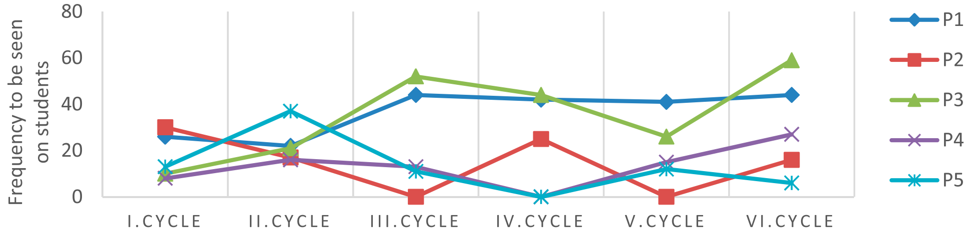

Figure 4. Findings regarding observations of the development of the sub-skill of reaching the right solution and proving it. * P: Proving.

Comparing their solutions with each other helped students develop their understanding and reaching the right solution:

Teacher: The students corrected their mistakes faster by working collaboratively in the group and by comparing their ways of finding solutions. This improved their ability to reach the right solution.

For deep understanding of mathematics, it is suggested that teachers provide opportunities for young learners to compare their ways of finding solutions and explain their reasoning to each other (Figure 4).

Indicators of this sub-skill were reaching the right solution (P1), prediction (P2), explaining the basis of the result in detail (P3), determining the operational steps accurately (P4) and understanding the difference between prediction and the right solution (P5). It was observed that the frequencies of reaching the right solution (P1) and explaining the basis of the result in detail (P3) improved. Understanding the difference between prediction and the right solution developed in the cycles in which they worked within the action plan. However, the skill of understanding the difference between prediction and the actual solution seemed to develop in a later cycle, indicating that the skill of prediction developed first and provided the basis for developing skill in understanding the difference between prediction and reality in the next cycle. Understanding the difference between prediction and reality aided the skill of reaching the right solution.

- *Student 1: . . . I did not used to have predictions close to real size of, for example, the table in our house or to the real cost of houses and cars, but now I have answers close enough.*
- *Student 5: Now, I am able to guess the height of big trees I see when I am with my family at the balcony.*
- *Student 9: When I had had some money to go to the market for shopping, I used to think so long to calculate total price of the things I was about to buy. Now, I can buy the things much faster.*

Explaining the reasons for solutions in (P3) was one of the students' greatest weaknesses identified during the observations before the study, and proper provisions were taken. Thus, it was one of the skills that developed regularly during the implementation. Because new topics were introduced only within the fifth cycle, which led the students to ask questions and the teacher to give explanations, the students could explain fewer solutions with their own reasons. This situation was associated with the theme of understanding and confirmed that the findings were consistent.

- *Student 8: ... one day, three different opinions arose in our group. Two of us couldn't convince a friend that his solution was incorrect. We got help from our teacher. He told us how to convince him. Finally, we were able to explain to him that his way of solution was not correct.*
- *Student 9: ... I did not used to have much self-confidence. I constantly used to think that my friends gave right answers, and I was wrong. But now, I can convince them by explaining my way of solution.*
- *Student 15: The lesson was so fun today. My friend gave such logical explanations that all of us were convinced.*
- *Student 9: Among two solutions, both are correct, but one is more logical, and I can prove that.*

The effort to reach the right solution and to prove it helped the students develop their understanding skill (U1).

Teacher: ... at the beginning, they used to choose one of the solution ways randomly like in multiple-choice exams ... Then they noticed how to do calculations, which meant how to think, and they have started to explain the way of solution they chose with proofs after their calculations.

During the implementation, the teacher started to focus more on understanding and proving rather than on reaching the right solution.

Teacher: I accepted the students' rounding answers to let them think flexibly. I gave importance to their ways of solution and the proofs they provided, rather than soley to the solution.

Not all students who reach the right solution can prove it. Therefore, proving should be prioritized for critical thinking and deep understanding in mathematics (Figure 5).

Indicators of the sub-skill of suggesting a new solution were designing a solution model (S1) and not being dependent on a single solution (S2). Designing a solution model here

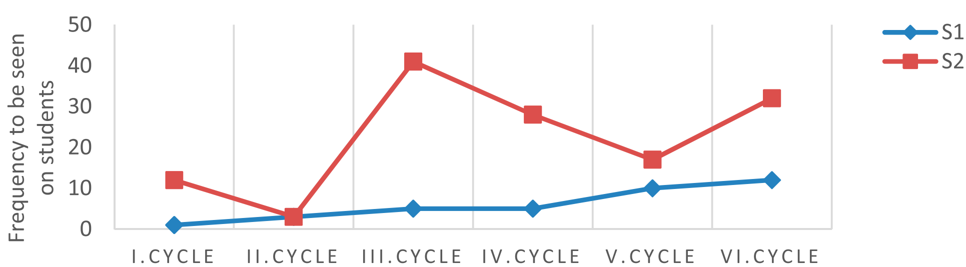

Figure 5. Findings regarding observations of the development of the sub-skill of suggesting a new solution. * s: suggesting a new solution.

referred to reaching the solution with an original interpretation, and not being dependent on a single solution referred to acknowledging multiple ways of reaching a solution. It was found that designing a solution model, although observed less frequently than not being dependent on a single solution, increased gradually, indicating that original solutions can be obtained when more students understand the process well. Hence the drop in frequency of S2 in the fifth cycle could be related to the students' having difficulty with authentic scenarios involving the concept 'time' within this cycle and to the teacher's struggle to help them by focusing on this content rather than their mathematics learning processes. In the sixth cycle, the students' learning issues had been determined and were accommodated within the frame of the action plan. Thus, during times when the students had deficient prior learning, their development of deep understanding weakened, and when authentic learning practices were patiently applied, their development accelerated in the next cycle:

- *Student 15: When I went shopping to buy shoes, I didn't used to understand when they said the shoes are 15% off. Now I can calculate it by myself by using different ways of solution.*
- *Student 12: . . . I used to think math was only about addition or subtraction, yet this year we are suggesting solutions using different operations. Thus, I have learned using different things in this lesson, and I started to use them in real life and my interest in math has increased . . .*
- *Student 9: I wasn't able to add up the amount of the fractions when I bought chocolate at the supermarket. I did not care about them at all. Now, I am aware of them, and I can calculate them instantly.*

The students had opportunities to discover different ideas as a result reading of real-life-based mathematical scenarios:

Teacher: The students got the opportunity to offer new ideas thanks to this teaching. The connection with the outside world helped them apply mathematics to real life and discover its uses. Since their observation skills were high, they had the chance to produce more different ideas.

Thus, applying authentic learning principles increased the students' frequencies of the behaviour of suggesting new solutions (Figure 6).

The sub-skill of reflection included the behaviours of suggesting personal ideas about one's own learning process (R1) and reflecting on the knowledge gained (R2). Interpretations of one's own learning referred to the student's thinking about and self-assessment of his/her learning. Did he/she understand the topic deeply? If not, why? While the frequency of such reflections was low in the earlier cycles, in the third cycle, in which the students were asked to assess in-class activities with their teacher and self-assess their learning, the frequency rose. The students exhibited R1 behaviour less frequently than R2 behaviour, the main reason for which was that the standards of authentic learning (depth of knowledge, connection with the outside world) gave more opportunity to reflect on their own learning rather than the learning process. Furthermore, the students were asked to keep a diary for improving their sub-skill of reflection, which enhanced their ability to formulate their observations of and personal opinions about the learning environment. Samples of reflections from students' diaries were as follows:

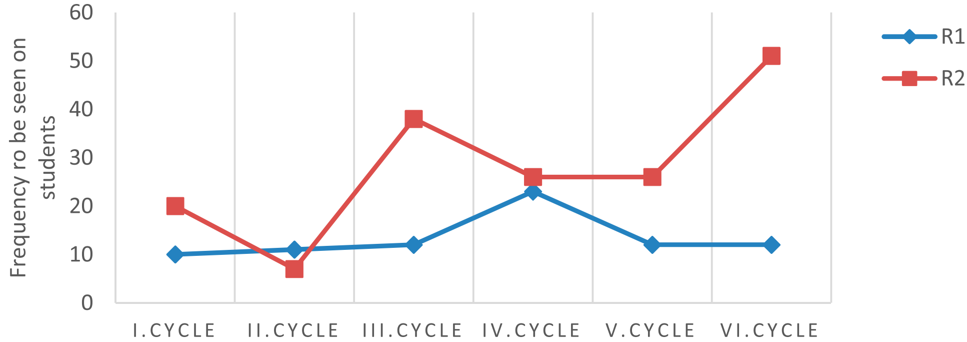

Figure 6. Findings regarding observations of the development of the sub-skill of reflection. * R: Reflection.

- *S3:Our teacher used not to call on the ones who didn't want to go to the board, but now he calls on them and gets their opinions (06.10.2014).*
- *S12: I am a shy person and I could overcome this matter with the help of this study (06.10.2014).*
- *S6: . . . I like this lesson. I used to speak without considering, but now I put thinking into it. My ability to convince increased regarding finding a solution . . . I was able to find the solution when I was given a scenario. I couldn't persuade my peers at the beginning, and I started to do it in time.*

Thinking about one's own learning and assessing the process are important subcomponents of critical thinking skills. To develop their reflections skills, students should be given chances to assess the teaching and learning process following each cycle so their reflections can influence the next phase.

4.2. Findings from the Rubric in relation to the development of critical thinking skills

In order to determine the level of development of fifth graders' critical thinking skills through application of authentic learning standards in math classes, the rubric for assessing critical thinking skills (CTR) included open-ended questions about the scenarios involving real life problems and related mathematical knowledge, which were analyzed after each cycle in relation to the sub-skills of critical thinking (see Appendix 1)The results of rubric evaluation for measuring critical thinking skills were presented in Figure 7.

It was inferred from the results of the open-ended evaluation administered to the students after each cycle that applying authentic learning principles in mathematics classes supported students' development of understanding, comparing, proving, suggesting a new solution and reflection skills. Although the skills were measured separately, their interrelatedness is clearly visible in Figure 7. Critical thinking starts with understanding (U) and continues with comparison and evaluation of solutions (C), reaching the right solution and proving it (P), suggesting a new solution (S) and reflection (R). The spiral structure of the figure is evidence that the sub-skills are associated with each other. In particular, U develops in parallel with C and P. It can be inferred from this result that learning activities related

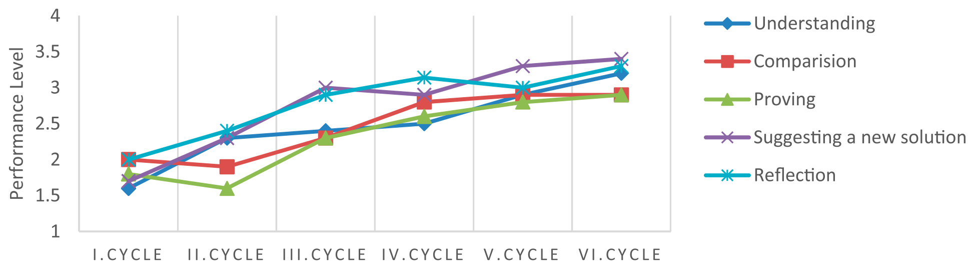

Figure 7. Results from the Rubric evaluation to measure critical thinking skills.

to proving and comparing can aid understanding of mathematics, as findings from both observations and rubric assessments indicated that development of C and P can be handled quite thoroughly, and that these processes directly support understanding in mathematics. Additionally, it can be suggested that S and R skills support each other because offering new solution is based on an individual's assessment of his/her own learning, which may directly reinforce reflection skills. This development is consistent with the observational results of the current study.

When working in groups and exposed to the higher-order thinking principles of authentic learning, students' abilities to understand a problem, compare understandings, and prove solutions developed. Additionally, based on the principle of depth of knowledge, frequent repetitions of knowledge gained through scenario problems and out-of-class connections might have supported the development of all dimensions, especially C and P. Moreover, the practuce of idea-sharing through meaningful dialogues and listening to each other and social support expressed in efforts to be respectful might have improved the students' critical thinking skills. Overall, the analysis of the rubric showed that development of the five sub-skills increased as the authentic learning principles became normalized as classroom culture, which reached its highest level in the sixth cycle.

5. Discussion

This study demonstrates that the contribution of authentic learning practices to developing critical thinking skills (Butera et al., 2014; Dennis & O'Hair, 2010; Glazer, 2001; Kitchener, 2008; Marcut, 2005; Miri et al., 2007; Preus, 2012) is as important in mathematical learning as in other disciplines. The constant development of the sub-skills of critical thinking that support mathematics learning, including understanding, comparison and evaluation of solutions, suggesting new solutions, and reflection, was observed all throughout the process.

The first finding to be discussed was that in this study it took the teacher more time to integrate authentic learning standards into the classroom than for the students to adapt. The first cycle largely reflected the teacher's struggle to change his traditional teacher-centered habits, a finding for which there is ample support in the literature. In mathematics instruction, both the students' and the teachers' personal traits are important (Schwartz, 2006). In their study examining the in-class practices of five teachers who claimed to be conducting authentic learning-based instruction, Dennis and O'Hair (2010) found that although all teachers had equal training and opportunity to apply the instruction, their application skills varied. Similarly Saleh (2009) concluded that the most crucial factor in the development of students' critical thinking skills are the teacher's skills and practices recommended that mathematics education programs should focus on the development of children's thinking skills. Interestingly, Innabi and El Sheikh (2006) reported that according to the literature on this issue, although most teachers claimed that they had to teach critical thinking, more than half did not use any strategies to promote critical thinking in their mathematics classes.

In the first cycle, the students generally tended to aim for the right solution rather than discuss the comprehension questions, which can be associated with the teacher's initial failure to use question types which would reinforce critical thinking but rather steered the students to find the right solution by using short-answer or multiple choice questions. Subsequently, he adhered more to the higher-order thinking and in-depth knowledge principles of authentic learning and started to lead them towards comparing and proving solutions. Alkhateeb (2019) found that the questions used by an eighth-grade mathematics teacher were not at a level sufficient for evaluation of critical thinking skills but were generally knowledge-based questions that encouraged memorization in order to reach the right solutions rather than fostering students' understanding. In general, there has been consensus that instruction supporting critical thinking skills should involve problems that that are meaningful to students, emphasize problem-solving processes over reaching right answers, and create a learning environment with sufficient opportunities for practice, generalization, proving and evaluation (Glazer, 2001).

Understanding is one of the sub-skills of critical thinking within an action plan based on the principles of authentic learning. Without developing critical thinking skills, students cannot deeply understand problems and problem-solving, and an important indicator for evaluating their critical thinking is how they expressed their understanding of their readings in their own words (Enright & Beattie, 1992). Butler (2019), arguing that instructional strategies for teaching concepts and reading comprehension techniques in mathematics are similar to those in other disciplines, called for further studies of mathematical understanding to address a gap in the research literature. In this study, repetitions of content were included to foster deep understanding, which is a principle of authentic learning, resulting in increased frequency of this aspect of student behaviour.

The teacher's balancing of attention between content and teaching method is important for fostering students' critical thinking. If the teacher focuses on content rather than authentic learning principles as in the first cycle, memorization is emphasized and the development of understanding skills is weakened (Aizikovitsh & Amit, 2010; Butler, 2019; Peter, 2012). Ediger (2009) also argued that mathematics teachers should be equally concerned about instructional methods and subject knowledge. The authentic learning scenarios provided in this study were compatible with the concurrent course content. Such connection with the real world supports students' understanding of the mathematical content (Forman and Steen, 1995). Examination of an exemplary lesson in this study revealed that it provided support for students' mathematical understanding and their evaluation and proving of solutions. Additionally, in-class problems which were consistent with real-life issues and concerns (e.g. ordering pizza, the size of the coin in their hands, the length of the desk they sat at, the national debt, etc.) and fostered a thinking culture in which the value of problems could be considered as students' attention was drawn to meaningful situations and they started to observe their environment more carefully. This greater alertness may have developed such skills as reading critically and with curiosity. Children are naturally curious about their environment, and this curiosity provides a basis for using their critical thinking skills and reasoning to make and check guesses (Boucher, 1998). According to Butler (2019), teachers' pay more attention to mathematical content and ability than to preparing students for real life uses of mathematics, or they teach real-life examples with shallow connections to mathematics learning and without increasing their reading skills and.

In the current study, in accordance with the principles of authentic teaching, students frequently connected problems with the outside world, listened to peers' opinions, and negotiated disagreements. Boucher (1998), who considered children's discussions of different kinds of ideas as opportunities to develop their critical thinking skills, offered ideas supporting this finding. According to Vygotsky, the language developed through discussion may increase students' knowledge of mathematics (Kocak et al., 2009). Saleh (2009) found that verbally expressing their mathematics knowledge fostered students' use of their observation abilities, which could improve their thinking skills. As mentioned above, in this study, students' discussions of the scenarios in their own language with peers, listening to each other, and using their reading and writing language skills in the classroom may have helped develop their critical thinking skills. Butler (2019) found that when reading, writing and listening activities were not associated with memorization, students' learning of mathematical subjects with understanding improved, which is the most significant aspect for the development of critical thinking.

The combination of both reaching and proving right solutions is a central component of learning mathematics (Ball et al., 2003; Reid, 2005). As authentic learning practices emphasize synthesizing information, reaching a valid solution, and explaining it (Preus, 2012), this dual skill improved in the current study. Although it has often been emphasized that understanding definitions and theorems and proving solutions are essential in mathematics learning, how proving is understood, especially in the fourth and fifth grades, and the role of the teacher in teaching it are unclear. In his study about proving in primary school mathematics, Stylianides (2007) emphasized the importance of the teacher's role. In addition, it is emphasized in the literature that proving, one of the sub-skills of critical thinking in mathematics, develops learners' conceptual and critical thinking skills (Reid, 2005). Through six cycles in this study, the students discussed ways of finding solutions with their peers and tried to convince each other through proving, which gave them the opportunity to discover new solutions by synthesizing their knowledge. Suggesting new solutions and not only depending on the present solutions are important for the development of critical thinking skills, yet they are challenging for the students. Firdaus et al. (2015), who tried to enrich the meaning of knowing and learning in middle school mathematics courses, concluded that activities featuring problems that led to social interaction in open environments were more effective than traditional mathematical activities for developing the skills of producing different solution theories and of justification. After observing five primary school teachers who applied group work in their lessons in order to improve the students' ways of thinking in mathematics, Cengiz et al. (2001) concluded that group work was effective for encouraging students to produce ideas and in develop alternative ways of finding solutions.

In the current study, the students showed improvement in expressing their thoughts about their process of learning and developing their reflection skills throughout the implementation. Reconsidering and reflecting on the assumptions behind one's beliefs are indicators of the development of critical thinking skills (Marcut, 2005), and the time allocated to students for social interaction and thinking in this study, in accordance with authentic learning standards, gave them opportunities to think about their learning and to engage in self-assessment. Similarly, according to Wilson and Wing Jan (1993), a reflective educational environment gives students chances to explain their thoughts and plan goals freely. Furthermore, an environment enabling them to assess their solutions mutually and suggest new solutions may give them the opportunity to interpret their learning. According to Tang (2000), students need to question their experiences and what they hear and reflect on ideas. Reasoning is a fundamental skill of critical thinking (Aizikovitsh & Amit, 2010; Marcut, 2005: Peter, 2012). The present study revealed that the students continued to think about and question what they had learned outside the classroom, which also may have promoted the development of their critical thinking skills.

Consequently, it can be asserted that mathematics learning can provide opportunities to develop critical thinking. At the same time, critical thinking is also an opportunity to improve mathematical learning. One of the ways to use this opportunity for mutual effectively is to employ instructional models that include principles of authentic learning, which emphasize that mathematics is a decision-making-based subject which gives importance to logical thinking and explicit expression of assumptions (Marcut, 2005). Similarly, according to Schwartz (2006), mathematics should be regarded as a subject that can help students develop the analytical and critical thinking skills required to solve real life problems, not just as a school course to be passed. The students' interest in the problems they encounter in their own lives may help develop their analytical perspectives and critical thinking skills. Preus (2012) stated that open-ended questions, short texts, visual resources, problem-solving assignments, discussion, and one-to-one explanations are elements of authentic learning-based applications, and that these practices foster higher-order thinking skills. From this point of view and supported by the current research findings, it can be suggested that further studies need to be conducted regarding the development of understanding, comparison and evaluation of solutions, proving, suggesting a new solution and reflection in mathematics learning. Also, in-class mathematics scenarios and open-ended questions that can develop these dimensions should be used at all levels of mathematics instruction.

On the other hand, it is difficult to assess and measure the development of critical thinking skills. Accordingly, the current study presented two important tools for monitoring the development of critical thinking skills in mathematics. One of these was the student behavioural indicators applied during observations and calculating the frequencies of these behaviours, and the other was the assessment rubric featuring open-ended questions. These tools may be shared in teacher inservice or pre-service teacher programs in the field of mathematics education.

6. Conclusion

Critical thinking is a necessary skill for individuals throughout their lives, and it is important to foster it in mathematics instruction as in all fields. The foremost result of this study was that the authentic learning model suggested by Newmann and Weglage (1993) improved fifth grade students' critical thinking skills of understanding mathematical knowledge, comparing solutions, reaching the right solution and proving it, suggesting a new solution and reflecting. Improvement of these skills is crucial for mathematical learning. Indeed, mathematics learning provides particular opportunities for the development of critical thinking skills. Hence, learning models that are appropriate for improving critical thinking should be included in mathematics curriculum planning and implemented in mathematics learning settings. Problems that entail repetition of knowledge involved in mathematics ought to be favoured and classroom scenarios properly connected with real life situations to develop students' critical thinking and in-depth understanding in mathematics. In addition, mathematics teachers and teacher candidates need to be educated in the teaching of critical thinking skills. This education ought to be structured to offer practice and reflection based on activities that foster learning processes rather than static knowledge acquisition.

Other implications regarding the research process can be listed as follows:

(1) It was found that the process of developing critical thinking skills was not the same in all cycles. It was also revealed that when prior learnings were incomplete or new learnings were in progress, this process was slower, suggesting the need for further studies. In this context, the instructor is recommended to be patient, to give the students enough opportunity to speak, to provide them with learning experiences until the development is enabled, and to give them feedback on their progress.
(2) It was found that improvement of teachers' attitudes and skills related to authentic learning practices was not realized in the first cycle, and arrangements were made for further skill development in this area. In light of this outcome, greater frequency of assessments of the teacher's learning process in the first cycle is advised.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendices

Appendix 1. Open-ended question example is presented below

Average house price in Antakya is around 170 K TL. Of how much a money machine approximately can pay a loan of 170 K TL house with 1 TL coins that it produces non-stop 24 hours a day (1 coin per second)?

Solution 1:

It can pay almost all the loan of 170 K house?

Solution 2:

It can pay about half of the loan of 170 K house?

Answer the questions about this scenario below.

(1) What are you asked with this scenario? Explain.
(2) Compare the two solutions given to you and state the one you think right. Explain the solutions by evaluating:If the first solution is right or wrong, why?If the second solution is right or wrong, why?
(3) Choose the solution or decision you see most appropriate for this situation and prove your choice by mathematical operations or a diagram.
(4) Suggest a new comparison example to develop the scenario.
(5) a) Reflect your own ideas on the issue.b) What did you learn with this scenario.

Appendix 2. Critical Thinking Skills Rubric (CTR)

Performance indicators	Scoring	Level 4 4 points	Level 3 3 points	Level 2 2 points	Level 1 1 points
I. Understanding the problem (identification)		Expresses the main idea and problem logically and consistently with their own sentences	He expresses his main idea and problem in a correct way by using more questions.	He expresses his main idea and problem in a weak way, with little detail.	Cannot determine the main idea and problem
II. Comparison and evaluation of solutions		Compares the solutions given with induction and deduction and determines the relevant information in an accurate manner.	Uses reasoning in evaluating solutions, achieve the right solution, but show some confusion in delivering relevant information.	Makes versatile mistakes in choosing information or evidence about solutions.	Cannot reveal information about solutions.
III. Reaching the right solution and proving it		Solves the problem clearly and with the correct process steps. It accurately describes the solution he has defended and the information base in full detail.	Solves the problem with the correct process steps. Explains some aspects of the information that the solution is based on.	He solves part of the problem, but he cannot get the results. Provides irrelevant explanations.	Cannot solve and defend the problem.
IV. Suggesting a new solution		Associates different information and ideas in a correct way and reveals a different way to achieve the right result.	Discovers different information and ideas. However, it cannot clearly reach a new conclusion.	Formats different information and ideas missing. He can't get the right result.	Cannot offer a new way or information.
V. Reflection		He / she is aware of the information he / she has learned individually and presents them with meaningful interpretations.	Identifies the information he has learned, but presents it superficially.	Has comments, but cannot provide the correct information.	Does not have an idea for individual thinking.