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Additional Heisenberg uncertainties ratio in the (1+3)Minkowski space and in the (1+4)D extended space model

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Abstract: **The article considers the possibility of introducing an additional Heisenberg uncertainty relation, which can be introduced in the (1+3)D Minkowski space and in the (1+4)D 5-dimensional extended space model (ESM). In the ESM (1+4)D formalism, in addition to the usual coordinates T;X,Y,Z, the interval S already existing in Minkowski space is considered as the fifth coordinate. The fifth coordinate S in the ESM has a physical meaning of action. The work shows that it is possible to introduce one more additional Heisenberg uncertainty relation connecting mass and interval.**

Keywords: **Minkowski space, (1+4)D extended space model, Heisenberg uncertainty relations**

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Content

1. INTRODUCTION

In 1908, at a meeting of naturalists and doctors in Cologne, Hermann Minkowski read his famous report on the geometric foundations of the theory of relativity, entitled "Space and Time" [1]. In his report, Minkowski noted that: "The views on space and time that I intend to develop in front of you arose on an experimental physical basis. This is their strength. Their tendency is radical. From now on, space itself and time itself should turn into a shadow, and only some kind of connection of both should remain independent."..No one has ever observed," Minkowski said, "any place other than at some point in time, and any time other than in some place." Minkowski calls a point in space corresponding to a given moment in time a "world point", and the totality of all world points that can only be imagined, for short, a "world". Then any body that exists for some time in space will correspond to a certain curve – a world line. «...The whole world seems to be decomposed into such world lines," Minkovsky continues his speech...physical laws could find their most perfect expression as the relationship between these world lines." In (1+3)In the Minkowski dimensional space, three spatial coordinates *x, y* and *z* and one time-like coordinate *t* are connected by the relation (1):

$$(ct)^2 - x^2 - y^2 - z^2 = s^2, \tag{1}$$

in the Minkowski space, the ratio (1) also includes the fourth value of the *s*-interval. The interval *s* is preserved during Lorentz transformations in space $M(T;\vec{X})$.

It is known that the mass, momentum and energy of a free particle are related by the ratio [2].

$$E^2 - c^2 p_X^2 - c^2 p_Y^2 - c^2 p_Z^2 = m^2 c^4. \tag{2}$$

The mass of a free particle m in the ratio (2) is a constant value. The ratio (2) is an analogue of the ratio (1) in (1+3)*D* is a dimensional space $M'(E; p_X, p_Y, p_Z)$ conjugate to the space *M(t; x, y, z)*. In the work of Weiner Heisenberg, published in 1927 [3], the uncertainty principle was formulated. Heisenberg notes that his research was written as a consequence of the works that preceded the advent of quantum mechanics published by Bohr, Einstein, Pauli, de Broglie. In Heisenberg's original work, the uncertainty principle is formulated for the uncertainty of determining the coordinate and momentum of a particle. If the uncertainties of the coordinate and momentum are understood as the standard deviations of these physical quantities from the average values, then the non-distribution ratios have the form:

$$\Delta x \Delta p_X \geq \hbar,\ \ \Delta \mathrm{y} \Delta p_Y \geq \hbar,\ \ \Delta \mathrm{z} \Delta p_Z \geq \hbar. \tag{3}$$

For energy *E* and time *t*, the Heisenberg uncertainty relation is written as:

$$\Delta E \Delta t \geq \hbar. \tag{4}$$

As we can see, in the Heisenberg uncertainty relations (3) and (4), pairs of canonically conjugate variables from equations (1) and (2) are combined together. Equations (1) and (2) relate energy with momentum and mass in (1+3) Minkowski space, as well as time with coordinates and interval (in this case, the mass and interval in Minkowski space are assumed to be constant values).

Later, various aspects of the development of quantum mechanics were developed and described, for example, in [4-10]. The uncertainty principle applies not only to coordinate and momentum (as it was first proposed by Heisenberg). In its general form, it is applicable to every pair of conjugate variables. For each pair of conjugate quantities, there is a generalized uncertainty relation of the same kind: $\Delta A \Delta B \geq \hbar$, that is, for a given normalized function and two specified Hermitian operators A and B, this inequality can be established, Robertson [11,12]. Therefore, the term uncertainty ratio is often used in the plural (uncertainty ratios), both when it comes to uncertainty ratios in general, and in cases where several specific ratios are meant for different quantities, and not for just one pair.

2. GEOMETRIC APPROACH TO PHYSICS (1+3)D MINKOWSKI SPACETIME AND THE PYTHAGOREAN THEOREM

In Okun L.B. work "The Theory of Relativity and the Pythagorean theorem" [13] it is noted that in the theory of relativity, the experimentally verifiable ratio (2) between the mass of a body (particle), energy and momentum is, from a mathematical point of view, a ratio similar to the Pythagorean theorem. In this case, mass and momentum are "legs" and energy is a "hypotenuse" in such a triangle. "As Minkowski showed, the theory of relativity acquires the simplest form if it is considered in a four-dimensional space-time" [13]. In the theory of relativity, the energy and momentum of a body (particle) form a 4-dimensional energy-momentum vector. The mass is a Lorentz scalar characterizing the length of the 4-vector (5)

$$E^2 - c^2 p_X^2 - c^2 p_Y^2 - c^2 p_Z^2 = m^2 c^4. \tag{5}$$

"This 4-dimensional space is pseudo-Euclidean – hence the minus sign in the formula for the square of length" [13].

Accordingly, in the theory of relativity, the magnitude of time and the coordinates of the body (particle) also form a 4-dimensional vector of time coordinates. The interval S is a Lorentz scalar characterizing the length of the 4-vector (6):

$$(ct)^2 - x^2 - y^2 - z^2 = s^2, \quad (6)$$

It is interesting to note that if we consider relations (1) and (2) from a formal mathematical point of view, as an ordinary triangle in Euclidean space, i.e. "the square of the hypotenuse in a right triangle is equal to the sum of the squares of the legs" nothing in formula (6) will change. Thus, the 4-dimensional (1+3)D Minkowski space, including the relation (1), can be considered Euclidean from a mathematical point of view, as soon as we consider (ct) as a "hypotenuse", and the components x, y, z, s, respectively, as "cathets":

$$(ct)^2 = s^2 + x^2 + y^2 + z^2. \quad (7)$$

As can be seen from the comparison of the relations describing the relationship of time, spatial coordinates, interval; energy, momentum and mass of the particle (1), (2) and the Heisenberg uncertainty relations (3), (4) for the same quantities, the relations (3) and (4) relate only the coordinate-momentum and energy-time. If we formally consider relations (1) and (2) as relations obeying the Pythagorean theorem in 4-dimensional multidimensional "right triangles" between the "hypotenuse" and "legs" leading to the non-distribution relations (3) and (4), then this relationship should extend to the remaining and present in (1) and (2) values: mass M and interval S. In this case, the magnitude of this uncertainty will be of the same order as for ratios (3) and (4):

$$\Delta m \Delta s \geq \hbar. \quad (8)$$

The fact that we consider the mass and the interval in Minkowski space as Lorentz scalars when formally considering relations (1) and (2) from a geometric point of view does not in any way affect the formal conclusion about the uncertainty present in determining the quantities included in relation (8).

3. FIVE-DIMENSIONAL MODELS, COMPARISON WITH (1+4)D IS AN EXTENDED SPACE MODEL

In modern science, the problem of combining gravitational and electromagnetic fields into one is actively discussed. Historically, this topic has been discussed for more than 100 years. Most attempts to construct a model of unifying fields are realized by constructing geometric models of physical interactions and interpreting physics within the framework of geometry in spaces of a larger number of dimensions. A review of the literature on multidimensional theories can be found, for example, in [14].

At the very beginning of the 20th century, Felix Klein [15] considered the Hamilton-Jacobi theory as optics in spaces of a larger number of dimensions. Klein's ideas were not further developed at the time of creation. Interest in the problem of geometrization of physics was re-aroused by the creation of the General Theory of Relativity (GRT). Researchers have made efforts to describe electromagnetism in geometric terms by analogy with gravity. The authors of these works tried to expand in various ways the already created GRT scheme instead of creating new models. The models of Klein O. [16] and Kaluza [17] became famous. One can also mention the works of Fock [18] and Mandela [19]. Note that a 5-dimensional space was used to construct these models. However, no clear physical interpretation of the fifth coordinate has been made in these works. Further attempts to develop 5-dimensional models were made by many scientists, including Einstein [20], de Broglie, Gamow [14] and Rumer [21], however, some interesting results did not work out. In 1927, de Broglie noted that the hypothesis of a five-dimensional world should require further assumption that our human organs of perception are unable to perceive the fifth variable [22]. "The changes in this fifth variable completely escape our sensory perception. Thus, if two points in the universe have the same values of the four variables of space and time, and the values of the fifth variable *X*5 are different, then one is indistinguishable from the other for us. We are, as it were, embedded in our four–dimensional manifold of space-time, and everything we perceive is a projection of the points of the five-dimensional universe onto this four-dimensional manifold".

We believe that the reason for the relative failure of these approaches was the lack of new physical hypotheses and the basis for a formal generalization of existing models.

Although in Rumer's 5-optics, the fifth coordinate is introduced in the form of an action and a 5-dimensional space with a metric is considered (+,-,-,-,-), but Rumer does not consider any transformations in this space that would confuse the coordinate with the other four coordinates of the Minkowski space. Accordingly, in the conjugate 5-dimensional space with coordinates, the mass also remains constant and is not converted into energy and momentum. It is also necessary to note the theory of gauge fields, in which electromagnetism, gravity and other interactions are considered from a single geometric point of view [23].

In the model proposed by P. Wesson and co-authors [24], it was proposed to use mass (matter) as the fifth coordinate, additional to the temporal and three spatial coordinates. This choice seems illogical to us. First of all, it leads to difficulties in generalizing the energy-momentum tensor to a 5-dimensional space. In addition, it is unclear what the movements that convert length into mass will look like, and what value corresponds to the mass in the conjugate space. Of course, mass can be used as the fifth coordinate, but not in coordinate space, but in momentum space, i.e. as an additional value to energy and the three components of momentum. And in the coordinate space, the fifth coordinate should be another quantity that is conjugate to the mass. Such a value is known – it is an action.

The approach proposed in the Extended Space Model [25-28] is fundamentally different from all these and similar theories. The ESM is based on a physical hypothesis, which consists in the fact that mass (rest mass) and its associated magnitude – action (interval) are dynamic variables. The magnitude of these variables is determined by the interaction of fields and particles. In this regard, such a model is a direct generalization of the Special Theory of Relativity (SRT). In SRT, the interval and the rest mass of the particles are invariants, and they can vary in ESM. In particular, a photon can acquire mass, both positive and negative. This mass can appear and change due to electromagnetic interaction and generate gravitational forces. It is this circumstance that makes it possible to consider gravity and electromagnetism as a single field in the ESM. The extended space model is a new theory formulated in 5-dimensional spacetime. During its construction, some assumptions were made that take the ESM beyond the framework of special relativity and standard quantum mechanics. One of the consequences of these assumptions is the appearance of mass in photons and their localization. This makes it possible to take into account additional mechanisms in the processes of interaction of elementary particles.

The reason for the introduction of ESM was the realization of the fact that the well-known experimentally proven ratio linking the energy momentum and mass of a particle can be considered as 5-dimensional if we take into account the possibility of mass changes in physical processes.

We consider a generalization of Einstein's special theory of relativity (SRT) to (1+4) *D*-dimensional space, with a metric (+,-,-,-,-), where the fifth coordinate is the interval s, which has the physical meaning of the action S [25].

The basis for this generalization is that in SRT the masses of particles are scalars and do not change during their elastic interactions. It is well known that a photon can be considered a massless particle and described as a plane wave only in an infinite empty space. If a photon enters a medium or finds itself in a confined space (in a resonator or waveguide), then it acquires a non-zero mass.

The length of the covariant 5-vector corresponding to objects satisfying (1) in the ESM is always zero:

$$(ct)^2 - x^2 - y^2 - z^2 - s^2 = 0. \quad (9)$$

Unlike (1+3)*D* Minkowski spaces, where the particle is mapped to a 4-dimensional isotropic for massless particles and non-isotropic for massive particles 4-energy-momentum vector (*E/c; p(x), p(y), p(z)*), in (1+4)An isotropic 5-dimensional energy-momentum-mass vector (*E/c; p(x), p(y), p(z), mc*), covariant for all particles, whose length is always zero (2), is compared to an MPP and a massless and massive particle.

To transfer the description of physical processes from the model of point particles (1), (2) to the model of fields within the 5-dimensional ESM, a 5-dimensional vector potential is introduced instead of a 4-dimensional vector potential in 4*D* Minkowski space: $(\varphi, \vec{A}, A_S) = (A_T, A_X, A_Y, A_Z, A_S)$ and on its basis, an energy-momentum-mass tensor (5×5) is constructed instead of a tensor (4×4) energy-momentum in Minkowski space [25-28].

It is important to note that the formalism of the ESM construction allows us to reduce all new formulas and relations appearing in the ESM to the classical formulas and relations of mechanics, electromagnetism and special relativity by simply imposing the condition that the interval s (mass m) remain constant.

4. ADDITIONAL UNCERTAINTY RATIO WITHIN THE FRAMEWORK (1+4)D EXTENDED SPACE MODELS (ESM)

In (1+4)*D* ESM mass M and interval *S* are no longer Lorentz scalars, unlike space (1+3)*D* Minkowski and can change during rotations in the planes (*T;S*), (*X,S*), (*Y,S*) and (*Z,S*) [24-26]. These turns have a clear physical meaning and are absent in the Minkowski space. The mass *M* and the interval *S* are included in the corresponding ratios (1) and (2) as variables having the same dimension as the other terms of the ratios (1) and (2) - energy/momentum and time/coordinates, respectively.

The physical meaning of rotations in the planes (*T;X*), (*T;Y*), (*T;Z*) existing in space (1+3) *D* Minkowski and *B* (1+4)*D* ESM is an increase/decrease in the velocity of massive particles (bodies) and a change in frequency for massless particles. Rotations in the planes (*X,Y*), (*X,Z*) and (*Y,Z*) have in both (1+3)*D* and (1+4)*D* spaces, the physical meaning of instantaneous rotations or changes in the size of bodies . The physical meaning of the new turns (*T;S*), (*X,S*), (*Y,S*) and (*Z,S*) existing in only (1+4)*D* extended space can be clearly understood from the application of these rotations to the five-dimensional energy-momentum-mass vector (*E/c;p(x),p(y),p(z), mc*). As shown in [25-27], rotations in the plane (*T;S*) "mix" energy and mass in the five-dimensional energy-momentum-mass vector, but keep the momentum unchanged. Rotations (*X,S*), (*Y,S*) and (*Z,S*) "mix" the momentum and mass of the particle, but leave the total energy of the particle unchanged.

From the point of view of the formal geometric approach, massive and massless particles are located in the MCP on 4-dimensional hyperboloids (1) and (2) located in 5-dimensional space. Thus, in our opinion, one of the fundamental approaches to the construction of a physical frame of reference is practically implemented in the ESM, which consists in the possibility of reversibly converting potential energy into kinetic energy using a simple coordinate transformation and vice versa [29].

When constructing the ESM, the basic principles of the construction of the general theory of relativity are preserved: symmetries, conservation laws, equality of gravitational and inert masses, motion along geodetic lines, the principle of general covariance, least action, conservation of energy, causality.

As we can see, Heisenberg, in the uncertainty relations (3) and (4), combined together pairs of canonically conjugate variables from equations (1) and (2) linking energy-momentum-mass and time-coordinates-interval in (1+3) Minkowski space (in this case, mass and interval in Minkowski space rely constant values).

Note that in [30] the authors J.Overduin and RC.Henry published an article devoted to the study of the manifestation of the Pythagorean theorem through the relation (1) in (1+4)-dimensional space, which largely echoes the analysis of the connection between the Pythagorean theorem and the theory of relativity, published for (1+3)*D*-dimensional Minkowski space in [13].

Also, in the works of Wesson P. [31,32], which are formally closest to the ESM, the physical meaning of the fifth coordinate and the possible uncertainty ratio resulting from the transition from 4 to 5 dimensional space are discussed. However, there is no clear physical meaning of the interval as an action and the ratio of uncertainty in the form (8) in these works.

In (1+4)*D* ESM mass and interval (action) are variables in equations (1) and (2), so in (1+4)*D* ESM a new uncertainty ratio is added to the already existing uncertainty ratios (3) and (4), which coincides with the previously obtained ratio in (1+3)*D* Minkowski space:

$$\Delta m \Delta s \geq \hbar. \qquad (10)$$

We can estimate the magnitude of the variable change from the following simple considerations:

1) For free massless particles propagating in empty space, the value of the interval S, in accordance with the provisions of the special theory of relativity, is 0.

When entering an external field, a massless particle becomes unfree and begins to interact with it, and we associate its propagation velocity in MP with the interval S through the refractive index n, which characterizes the ratio of the velocity of a free particle in empty space with the velocity of a non-free particle in an external field (regardless of the physical nature of this field) [25-27,33] (in ESM, this case formally corresponds to a combination of rotations in the planes (TS)+(XS)).

Thus, we will be able to estimate the magnitude of the interval change when a massless particle enters the external field from empty space.

2) For free massive particles, another approach can be used - the work of L.B. Okun [13] provides an estimate of the neutrino mass oscillation $\Delta m_{12}^2 = (0.8 \pm 0.04) \cdot 10^{-4} eV^2$; $\Delta m_{32}^2 = (25 \pm 6) \cdot 10^{-4} eV^2$.

Without going into the physical cause of the neutrino mass oscillation, we formally substitute these estimates of the mass oscillation into a new uncertainty ratio (10) to obtain estimates of the ΔS magnitude corresponding to the neutrino mass oscillation.

5. CONCLUSION

Presented in the work (1+4)The extended space model is based on the physical hypothesis that mass (rest mass) and its associated magnitude – action (interval) are dynamic variables. The magnitude of these variables is determined by the interaction of fields and particles. The extended space model is a direct generalization of the Special Theory of Relativity (SRT).

In SRT, the interval and the rest mass of the particles are invariants, and they can vary in ESM. Based on the assumptions made in the ESM, the extended Maxwell equations are constructed, which, along with the electromagnetic interaction, also describe gravitational effects, and also have a structure similar to the relativistic generalization of the Schrodinger equation in field-free space in the form of the Klein-Gordon equation.

Heisenberg uncertainty ratios in (1+4)*D* space taking into account the assumptions accepted in the ESM, they also acquire an expanded form and a new ratio is added to the already existing ratios (3) and (4) in the ESM, which has the same form in (1+3)*D* and (1+4)*D* spaces: (8), (10).

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